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On Presic Type Generalization of the Banach Contraction Mapping Principle in Multiplicative Metric Spaces

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Abstract: In this paper we prove a unique common fixed point theorems using Presic type contraction in complete multiplicative metric spaces.

Keywords: Multiplicative metric spaces, Presic type contraction, k-weak compatible mappings, fixed point.

Mathematics Subject Classification: 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

The set of positive real numbers is not complete with respect to usual metric. To overcome this difficulty, in 2008, Bashirov et al. [5] introduced the concept of multiplicative metric spaces as follows:

Definition1.1. ([5]) Let X be a non-empty set. A multiplicative metric is a mapping $d: X\times X \to \mathbb{R}^+$ satisfying the following conditions:
(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x=y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z).\, d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).
Then mapping d together with X i.e., (X, d) is known as multiplicative metric spaces.

Example1.2.([5]) Let R^n_+ be the collection of all n-tuples of positive real numbers.
Let $d^*: \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defined as follows:

$$d^*(x, y) = \left(\left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* \ldots \left|\frac{x_n}{y_n}\right|^*\right),$$

where $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n) \in \mathbb{R}^n_+$ and $|.| : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a|^* = \begin{cases} a & if\ a \geq 1; \\ \frac{1}{a} & if\ a < 1. \end{cases}$$

Then (X, d) is a multiplicative metric space**.**

Example1.3. ([10]) Let $d: \mathbb{R} \times \mathbb{R} \to [1, \infty)$ be defined by
$d(x, y) = a^{|x-y|}$,where $x, y \in \mathbb{R}$ and $a > 1$. Then d(x, y) is multiplicative metric and (X, d) is a multiplicative metric space. We may call it usual multiplicative metric spaces.
In 2015, M. Abbas et.al. introduced the notion of multiplicative absolute value function as follow:

Definition 1.4.([2]) A multiplicative absolute value function $|\cdot|: \mathbb{R} \to \mathbb{R}^+$ is defined as

$$|x| = \begin{cases} x & if \quad x \geq 1 \\ \frac{1}{x} & if \quad x \in (0,1) \\ 1 & if \quad x = 0 \\ -\frac{1}{x} & if \quad x \in (-1,0) \\ -x & if \quad x \leq -1 \end{cases}.$$

Proposition 1.5.([2]) For arbitrary $x, y \in \mathbb{R}^+$,the multiplicative absolute value function $|\cdot| : \mathbb{R}^+ \to \mathbb{R}^+$satisfies the following:
(1) $|x| \geq 1$.
(2) $x \leq |x|$.

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(3) $1/|x| \leq x$ if $x > 0$ and $x \leq 1/|x|$ if $x \leq 0$.
(4) $|x \cdot y| \leq |x||y|$.
One can refer to ([10]) for detailed multiplicative metric topology.

Definition1.6.([7]) Let (X, d) be a multiplicative metric space. A sequence $\{x_n\}$ in X said to be a
(i) multiplicative convergent sequence to x, if for every multiplicative open ball
B_ϵ(x) = { y | d(x, y) < ϵ} , ϵ > 1, there exists a natural number N such that $x_n \in B_\epsilon$(x) for all
n ≥ N, i. e, $d(x_n, x) \to 1$ as $n \to \infty$.
(ii) multiplicative Cauchy sequence if for all ϵ > 1, there exists N ∈ ℕ such that $d(x_n, x_m) < \epsilon$ for all m, n > N i. e , $d(x_n, x_m) \to 1$ as $n \to \infty$.
A multiplicative metric space is called complete if every multiplicative Cauchy sequence in X is multiplicative convergent to x ∈ X.
In 2012, Ozavsar gave the concept of multiplicative contraction mapping and proved some fixed point theorem for these maps in complete multiplicative metric spaces.

Definition1.7.([7]) Let (X, d) be a multiplicative metric space. The map f : X → X is called a multiplicative contraction if there exists a real constant λ ∈ [0, 1) such that
$d(f(x_1), f(x_2)) \leq (d(x_1, x_2))^\lambda$ for all x, y ∈ X.
Consider the k-th order nonlinear difference equation

$$x_{n+k} = f(x_n, ..., x_{n+k-1}),\ n \in \mathbb{N} \tag{1.8}$$

with the initial values $x_0, x_1, ..., x_k \in X$, where (X, d) is a metric space, k ∈ N, k ≥ 1
and f : $X^k \to$ X. Equation (1.1) can be studied for fixed point theory in view of the fact that
$x^* \in$ X is a solution of (1.1) if and only if x^* is a fixed point of f , that is, $x^* = f(x^*, ..., x^*)$.

Definition 1.9. Let (X, d) be a metric space, k a positive integer, and
$f: X^k \to X$ and $g: X \to X$ mappings.
(b) An element $x \in X$ is said to be a fixed point of f if x = (x,...,).
(c) If $x = gx$ = f(x,...,x),then x is called a common fixed point of f and g.
(d) Mappings f and g are said to be commuting if ((x,...,)) = $f(gx,...,gx)$, for all $x \in X$.
(f) Mappings f and g are said to be weakly commuting if
d(f(g(x, x, ...x)),g(fx, fx, ...fx)) ≤ d(f(x, x, ...x),g(x, x, ...x)) for all x ∈ X.
(b) An element $x \in X$ is said to be a coincidence point of f and g if gx = (x,...,).
(e) Mappings f and g are said to be k-compatible (coincidentally commuting)
if g (f(p, p, . . . , p)) = f(gp, gp, . . . , gp), whenever p ∈ X is such that gp = f (p, p, . . . , p).

Remark 1.10. The above definition are used in similar mode multiplicative metric spaces.

Remark 1.11. For k=1, the above definitions reduce to the usual definition of commuting and weakly compatible mappings in a multiplicative metric space.
In 1965, S.B. Presic in [8] gives the most important results on this direction by generalizing the Banach contraction mapping principle as follows:

Theorem 1.12. ([8]). Let (X, d) be a complete metric space, k a positive integer and
T : $X^k \to$ X a mapping satisfying the following contractive type condition
(1.2.1) $d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + ... + q_k d(x_k, x_{k+1})$ for every $x_1, x_2, x_3, ..., x_k, x_{k+1}$ in X, where $q_1, q_2, ..., q_k$ are non -negative constants such that $q_1 + q_2 + ... + q_k < 1$.
Then there exists a unique point x in X such that T (x, x, ..., x) = x.
Moreover, if $x_1, x_2, ..., x_k$ are arbitrary points in X and for n ∈ N , $x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1})$ then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, ..., \lim x_n)$.

2. MAIN RESULTS

In 1997, Alber and Guerre-Delabriere [4] introduced the notion of weakly contractive mappings in Hilbert spaces and proved that any weakly contractive mapping defined on complete Hilbert spaces has a unique fixed point. Rhoads [9] extended their work in Banach spaces.

Definition 2.5.[9] A mapping f : X → X is said to be a weakly contractive if

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y));\ \text{for all } x, y \in X;$$

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where φ : [0, 1) → [0, 1) is a continuous and non-decreasing function such that it is positive in (0,∞),φ (0) = 0 and $\lim_{t\to\infty}$ φ (t) = 0.
In 2015, M. Abbas, D. Ili´c , T. Nazir[3] ,proved following theorem in metric spaces as follows:

Theorem2.6.[3] Let (X, d) be a complete metric space, k a positive integer and f : $X^k \to$ X be a given mapping. Suppose that there exists φ: [0, ∞) → [0, ∞) a lower semi-continuous function with
φ (t) = 0 if and only if t = 0 satisfying
(2.7) T(X^k) ⊆ f (X),
(2.8) f (X) is complete and
(2.9) (f, T) is a weakly k-compatible pair.
(2.10) d(f ($x_1, x_2, x_3, \ldots, x_k$), f ($x_2, x_3, \ldots, x_{k+1}$))
$$\leq \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\} - \varphi(\max\{d(x_i, x_{i+1}),$$
1 ≤ i ≤ k}),
for all $(x_1, x_2, x_3, \ldots, x_{k+1}) \in X^{k+1}$.
(2.11) d(T (u, u, . . . , u), T (v, v, . . . , v)) < d(f u, f v), for all distinct u, v ∈ X.
Then, for any arbitrary points $x_2, x_3, \ldots, x_k$ ∈ X, the sequence $\{x_n\}$ defined by (2.7) converges to u ∈ X and u is a fixed point of f , that is, u = f (u,…,u).
Moreover, if d (f (x,…,x), f (y,…,y)) ≤ d(x, y) − φ (d(x; y));
holds for all x, y ∈ X with x ≠ y, then u is the unique fixed point of f .
Now we prove above theorem in setting of multiplicative metric space as follows:

Theorem2.12. Let (X, d) be a complete multiplicative metric space, k a positive integer and
T : $X^k \to$ X be a given mapping. Suppose that there exists φ: [1, ∞) → [1,∞) a lower semi-continuous function with φ (t) = 1 if and only if t = 1 satisfying
(2.13) T(X^k) ⊆ f (X),
(2.14) f (X) is complete and
(2.15) (f, T) is a weakly k-compatible pair.
(2.16) d(T($x_1, x_2, x_3, \ldots, x_k$), T($x_2, x_3, \ldots, x_{k+1}$))
$$\leq \frac{\max\{d(fx_i, fx_{i+1}) : 1 \leq i \leq k\}^{\lambda}}{\varphi(\max\{d(fx_i, fx_{i+1}) : 1 \leq i \leq k\}^{\lambda}}$$
for all $x_1, x_2, x_3, \ldots, x_{k+1} \in X^{k+1}$. Then, for any arbitrary points $x_2, x_3, \ldots, x_k$ ∈ X, the sequence $\{x_n\}$ defined by (2.13) converges to u ∈ X and u is a fixed point of T , that is, u = f (u,…,u).
(2.17) Moreover, if d (T (x,…,x), T (y,…,y)) ≤ d(fx, fy) − φ (d(fx,fy));
holds for all x, y ∈ X with x ≠ y, then u is the unique fixed point of f .

Proof. Let $x_1, x_2, x_3, \ldots, x_k$ be arbitrary elements in X. By (2.13), we define a sequence
$\{y_n\}$in f(X) as follows : y_{n+k}= f(x_{n+k})=T($x_n, x_{n+1}, \ldots x_{n+k-1}$), for n=1,2, ...
For simplicity set $\alpha_n = d(y_n, y_{n+1})$. We shall prove by induction that for each n ∈ N:
$$\alpha_n \leq K^{\theta^n} \quad (\text{where } \theta = (\lambda)^{1/k} < 1 ,\ K = \max\{(\alpha_1)^{1/\theta}, (\alpha_2)^{1/\theta^2}, \ldots, (\alpha_k)^{1/\theta^k}\} \qquad (2.18)$$
According to the definition of K we see that (2.18) is true for n = 1, . . . , k.
Now let the following k inequalities: $\alpha_n \leq K^{\theta^n}$, $\alpha_{n+1} \leq K^{\theta^{n+1}}$, . . . , $\alpha_{n+k-1} \leq K^{\theta^{n+k-1}}$ be the induction hypotheses.
Then we have:
$$\alpha_{n+k} = d(y_{n+k}, y_{n+k+1}) = d(T(x_n, x_{n+1}, \ldots, x_{n+k-1}), T(x_{n+1}, x_{n+2}, \ldots, x_{n+k}))$$
$$\leq \frac{\max\{d(fx_n, fx_{n+1}), d(fx_{n+1}, fx_{n+2}), \ldots d(fx_{n+k-1}, fx_{n+k})\}^{\lambda}}{\varphi(\max\{d(fx_n, fx_{n+1}), d(fx_{n+1}, fx_{n+2}), \ldots d(fx_{n+k-1}, fx_{n+k})\}^{\lambda}}$$
$$\leq \frac{[\max\{\alpha_n, \alpha_{n+1}, \ldots \alpha_{n+k-1}\}]^{\lambda}}{\varphi([\max\{\alpha_n, \alpha_{n+1}, \ldots \alpha_{n+k-1}\}]^{\lambda}}$$
$$\leq \frac{[\max\{K^{\theta^n}, K^{\theta^{n+1}}, \ldots K^{\theta^{n+k-1}}\}]^{\lambda}}{\varphi([\max\{K^{\theta^n}, K^{\theta^{n+1}}, \ldots K^{\theta^{n+k-1}}\}]^{\lambda}}$$
$$\leq [\max\{K^{\theta^n}, K^{\theta^{n+1}}, \ldots K^{\theta^{n+k-1}}\}]^{\lambda}$$
$$\leq [K^{\theta^n}]^{\lambda} \{\text{as } \theta < 1\}$$
$$= K^{\theta^{n+k}} \{\text{as } \lambda = \theta^k\}$$
Thus inductive proof of (2.18) is complete.
Now, for n, p ∈ N, we have
$$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}).d(y_{n+1}, y_{n+2}).....d(y_{n+p-1}, y_{n+p})$$
$$\leq K^{\theta^n}.K^{\theta^{n+1}}.....K^{\theta^{n+p-1}}$$

$\leq K^{\theta^n}(1+\theta+\theta^2+\cdots)$

$\leq K^{\theta^n}/_{1-\theta}$.

Letting $\to \infty$. Hence sequence $\{y_n\}$ is a Cauchy sequence in f(X). As f(X) is complete, there exists z ∈ f(X) such that $\lim_{n\to\infty} y_n = z$.

Hence there exists a point p ∈ X such that z = f p.

Now consider

$d(f x_{n+k}, T(p, p, \ldots, p)) = d(T(p, p, \ldots, p), T(x_n, x_{n+1}, \ldots, x_{n+k-1}))$

$\leq d(T(p, p, \ldots, p), T(p, p, \ldots, p, x_n)).d(T(p, p, \ldots, p, x_n), T(p, p, \ldots, p, x_n, x_{n+1}))$

$. d(T(p, p, \ldots, p, x_n, x_{n+1}), T(p, p, \ldots, p, x_n, x_{n+1}, x_{n+2})).d(T(p, p, \ldots, p, x_n, x_{n+1}, x_{n+2}), T(p, p, \ldots, p, x_n, x_{n+1}, x_{n+2}, x_{n+3})). \ldots . d(T(p, x_n, x_{n+1}, \ldots, x_{n+k-2}), T(x_n, x_{n+1}, \ldots, x_{n+k-1}))$.

$\leq \frac{[d(f p, f x_n)]^\lambda}{\varphi([d(f p, f x_n)]^\lambda)} . \frac{[\max\{d(fp, fx_n), d(fx_n, f x_{n+1})\}]^\lambda}{\varphi([\max\{d(fp, fx_n), d(fx_n, f x_{n+1})\}]^\lambda)}$

$\frac{[\max\{d(f p, f x_n), d(f x_n, f x_{n+1}), d(f x_{n+1}, f x_{n+2})\}]^\lambda}{\varphi([\max\{d(f p, f x_n), d(f x_n, f x_{n+1}), d(f x_{n+1}, f x_{n+2})\}]^\lambda)} . \frac{[\max\{d(f p, f x_n), d(f x_n, f x_{n+1}), d(f x_{n+1}, f x_{n+2})\}]^\lambda}{\varphi([\max\{d(f p, f x_n), d(f x_n, f x_{n+1}), d(f x_{n+1}, f x_{n+2})\}]^\lambda)} \ldots$

$\frac{[\max\{d(f p, fx_n), d(fx_n, f x_{n+1}), d(f x_{n+1}, f x_{n+2}), d(f x_{n+2}, f x_{n+3})\}]^\lambda}{\varphi([\max\{d(f p, fx_n), d(fx_n, f x_{n+1}), d(f x_{n+1}, f x_{n+2}), d(f x_{n+2}, f x_{n+3})\}]^\lambda)} . \frac{[\max\{d(f p, f x_n), d(fx_n, f x_{n+1}), \ldots, d(f x_{n+k-2}, f x_{n+k-1})\}]^\lambda}{\varphi([\max\{d(f p, f x_n), d(fx_n, f x_{n+1}), \ldots, d(f x_{n+k-2}, f x_{n+k-1})\}]^\lambda)}$.

Letting n → ∞, and using the properties of φ we get

d(fp, T (p, p, . . . , p)) ≤ 1, so that f p = T (p, p, . . . , p).

Since (f, T) is weakly k-compatible we have

f (T (p, p, . . . , p)) = T (f p, f p, . . . , f p)and sof^2p = f (f p) = f (T (p, p, . . . , p)) = T(f p, f p, . . . , f p).

Thus f z = T (z, z, . . . , z).

We now have

$d(f^2p, f p) = d(T(f p, f p, \ldots, f p), T(p, p, \ldots, p)) \leq \frac{d(f^2p, f p)]^\lambda}{\varphi(d(f^2p, f p)^\lambda)} < d(f^2p, f p)$, which is a contradiction.

Therefore, f^2p = f p so that f z = z. We now have z = fz = T (z, z, . . . , z).

Uniqueness can be easily found from (2.17).

REFERENCES

[1] M. Abbas, B. Ali, Y. I. Suleiman, Common fixed points of locally contractive mappings in multiplicative metric spaces with application, Hindawi Publishing corporation International Journal of Mathematics and mathematical sciences Volume 2015, article ID 218683.

[2] M. Abbas, M. De la Sen and T. Nazir, Common fixed points of generalized rational type cocyclic mappings in multiplicative metric spaces, Hindawi Publishing corporation Discrete Dynamics in Nature and Society, Vol. 2015, Article ID 532725,10 pages.

[3] M. Abbas, D. Ili´c , T. Nazir Iterative Approximation of Fixed Points of Generalized Weak Presic Type k-Step Iterative Method for a Class of Operators, Filomat 29:4 (2015), 713–724,DOI 10.2298/FIL1504713A

[4] Ya.I. Alber, S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, in: New Results in Operator Theory, in: I. Goldberg,Yu. Lyubich (Eds.), Advances and Appl., vol. 98, Birkhauser Verlag, 1997, pp.7-22.

[5] A.E. Bashirov, E.M. Kurplnara, and A . Ozyapici, Multiplicative calculus and its applicatiopns. J. Math. Anal. Appl. 337, (2008), 36-48.

[6] R. George, M. S. Khan, On Presic Type Extension of Banach Contraction Principle, Int. Journal of Math. Analysis, Vol. 5, 2011, no. 21, 1019 - 1024

[7] M .Ozavsar and A.C Cevikel, Fixed point of multiplicative contraction mappings on multiplicative metric space, ArXiv: 1205.5131v1 [matn.GN] (2012).

[8] S. B. Presic, Sur une classe d'inequations aux differences finite et sur laconvergence de certain es suites, Pub.de. l'Inst. Math. Belgrade 5(19)(1965), 75-78.

[9] B.E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Analysis TMA 47(4) (2001) 2683-2693.

[10] M. Sarwar, R. Badshah-e, some unique fixed point Theorems in multiplicative metric space, ArXiv: 1410.3384v2 [matn.GM] (2014).